ISRAEL JOURNAL OF MATHEMATICS 79 (1992), 257–279

STRONG QUADRATIC MODULES

BY

G. STROTH

Freie Universität Berlin, 2. Mathematisches Institut
Arnimalee 3, 1000 Berlin 33, Germany

ABSTRACT

For a Chevalley group G over a field of characteristic 2 we determine all irreducible modules V over $GF(2)$ such that $[V, R, Q] = 0$, where R is a long root group and $Q = Z_2(O_2(N_G(R)))$. As a corollary we obtain a classification of those irreducible modules admitting a quadratic fours group E which intersect a long root group nontrivially but is not contained in such a group.

Introduction

One of the main tools in the quite recent theory of parabolic systems is the knowledge of $GF(p)$-modules V for the groups involved admitting quadratically acting p-groups.

Let G be a finite simple group, V a $GF(p)G$-module, p a prime. We call V quadratic if there is a nontrivial p-subgroup E of G such that $[V, E, E] = 0$. For $p \geq 5$ the groups G possessing such a module have been studied by J. Thompson [Th]. For $p = 3$ results have been obtained by Ch.Ho [Ho]. The corresponding modules for the groups of Lie type have been classified by A.A. Premet and I.D. Suprunenko [PS]. Similar results have been obtained by U. Meierfrankenfeld [Mei]. If $p = 2$ the definition of a quadratic module is not any restriction at all as for any involution x we have $[V, x, x] = 0$. So in this case we have to add a further assumption. In [MeiStr1] and [MeiStr2] the case of G a sporadic simple group, an alternating group or a group of Lie type over a field of odd characteristic was

Received July 22, 1991 and in revised form January 28, 1992

investigated under the additional assumption that $|E| \geq 4$. Again this seems not to be the right condition for groups of Lie type over a field of characteristic 2, as root groups tend to act quadratically on many modules. So a classification, which might be possible, would contain a long list of modules defined over $GF(q)$ (if G is defined over $GF(q)$) and so would be just too long for interesting applications. This paper is a first step in obtaining a satisfactory result for quadratic modules for Chevalley groups over $GF(q)$, $q = 2^n$. We are going to prove the following theorem.

THEOREM: *Let $G = G(q)$ be a Chevalley group over $GF(q)$, $q = 2^n$. Let R be a long root group and $Q = Z_2(O_2(N_G(R)))$. Let V be a nontrivial irreducible $GF(2)G$-module with $[V, R, Q] = 0$. Then one of the following holds:*

(i) $G \simeq (S)L(n,q)$ *or* $(S)U(n,q)$ *and* $V = V(\lambda)$ *for some fundamental weight* λ.

(ii) $G \simeq \Omega^{\pm}(2n,q)$ *or* $Sp(2n,q)$ *and V is the natural module or a spin module.*

(iii) $G \simeq E_6(q)$ *and* $V = V(\lambda_1)$ *or* $V(\lambda_6)$.

(iv) $G \simeq E_7(q)$ *and* $V = V(\lambda_7)$.

(v) $G \simeq {}^2E_6(q)$ *or* $F_4(q)$ *and* $V = V(\lambda_4)$.

(vi) $G \simeq G_2(q)$ *or* ${}^3D_4(q)$ *and V is the natural module.*

In section 1 we prove the Theorem, and then in sections 2 and 3 we deduce the following corollaries. In fact corollary 1 had originally inspired the work of this paper.

COROLLARY 1: *Let $G = G(q)$ be a Chevalley group over $GF(q)$, $q = 2^n$, and V be a nontrivial irreducible $GF(2)G$-module. Let X be some nontrivial elementary abelian 2-subgroup of G such that $[V, X, X^g] = 0$ whenever $[X, X^g] = 1$, $g \in G$. Then V is one of the modules in the theorem or $G \simeq (S)U(3,q)$ and V is a basic module (8-dimensional module) or $G \cong Sz(q)$ and V is the natural module.*

A question like the one of corollary 1 recently occurred in a paper of Timmesfeld [Ti]. If E is a quadratic fours group we get the following corollary.

COROLLARY 2: *Let $G = G(q)$ be a Chevalley group over $GF(q)$, $q = 2^n$, and V be a nontrivial irreducible $GF(2)G$-module. Let E be a fours group such that $[V, E, E] = 0$. If E intersects some long root subgroup R of G nontrivially, but $E \not\subseteq R$, then either V is one of the modules in the theorem or $G \cong \mathrm{Sp}(2n,q)$ or $F_4(q)$ and $V = V(\lambda)$ for some fundamental weight λ.*

I do not know whether $G = F_4(q)$ and $V = V(\lambda_2)$ really occurs.

For a Chevalley group G let $q(G)$ be the minimum value of n such that the following is true: If V is an irreducible module and A is some elementary abelian subgroup of G with $[V, A, A] = 0$ and $|A| = 2^n$, then V is one of the modules in corollary 2. It would be interesting to determine $q(G)$.

0. Preliminaries

LEMMA 0.1: *Let G be a finite group and $\langle x, y\rangle \le G$ be a fours group acting nontrivially on some irreducible $GF(2)G$–module V. Let $x \sim y \sim xy$ and $G = \langle \mathbb{C}_G(s), \mathbb{C}_G(t)\rangle$ for any pair s, t from the triple x, y, xy such that $\langle s, t\rangle = \langle x, y\rangle$. Then $[V, x] \cap [V, y] \neq 0$.*

Proof: As V is an irreducible module we have $\mathbb{C}_V(x) \neq \mathbb{C}_V(xy)$. So

$$[\mathbb{C}_V(xy), x] = [\mathbb{C}_V(xy), y] \neq 0.$$

∎

LEMMA (0.2): *Let W be a 2–group which carries the structure of a 2–dimensional vectorspace over $GF(q)$, $q = 2^n$. Let V be a $GF(2)W$–module on which q of the $q + 1$ one-dimensional subspaces of W act quadratically. Then any one-dimensional subspace of W acts quadratically.*

Proof: Let $W = W_1W_2$ where $W_1 = \langle x_1, \dots, x_n\rangle$, $W_2 = \langle y_1, \dots, y_n\rangle$ and W_1, W_2 are one–spaces which act quadratically. We may assume $n \ge 2$.

Let $w_1, w_1' \in W_1$, $w_2, w_2' \in W_2$, $v \in V$. We first show

$$\begin{aligned}[v, w_1w_2, w_1'w_2'] &= [v, w_1, w_2'] + [v, w_1', w_2] \\ &= [v, w_2', w_1] + [v, w_2, w_1'].\end{aligned} \tag{1}$$

For this we use freely the following general facts

$[v, ab] = [v, a] + [v, b] + [v, a, b]$ and
$[v, a, b] = [v, b, a]$ for commuting a, b.

We have

$$z = [v, w_1w_2] = [v, w_1] + [v, w_2] + [v, w_1, w_2] = [v, w_2] + x = [v, w_1] + y$$

where $x \in [V, W_1]$ and $y \in [V, W_2]$.

Now

$$[v, w_1 w_2, w_1' w_2'] = [z, w_1' w_2'] = [z, w_1'] + [z_1, w_2'] + [z, w_1', w_2'].$$

We have

$[z, w_1'] = [v, w_2, w_1']$ as $[x, w_1'] = 0$ by quadratic action,
$[z, w_2'] = [v, w_1, w_2']$ as $[y, w_2'] = 0$ by quadratic action,
$[z, w_1', w_2'] = [[v, w_2], w_1', w_2'] = [[v, w_2], w_2', w_1'] = 0$ by quadratic action.

This establishes (1).

For proving the lemma it is enough to show that for $x_1 y,\ x_2 z \in W_0$, W_0 a one-space in W, $y, z \in W_2$, we get $[v, x_1 y,\ x_2 z] = 0$ for any $v \in V$.

Let $y = \prod y_i^{m_i}$ then set $\overline{y} = \prod x_i^{m_i}$. So to $x_1 y$, $x_2 z$ there correspond $y_1 \overline{y}$ and $y_2 \overline{z}$. Looking at W as a two dimensional vectorspace over $GF(q)$ implies that there is an automorphism ρ of order $q-1$ (the multiplication by field elements) which acts on W_1 and W_2 in the same way. Now the $q+1$ one dimensional subspaces of W are just the $q+1$ $\langle \rho \rangle$-invariant subgroups of order q. As $x_1 y$ and $x_2 z$ are in W_0 there is some power of ρ mapping x_1 onto x_2 and y onto z. But then the same power maps y_1 onto y_2 and $\bar{y}$ onto $\bar{z}$, so $y_1 \bar{y}, y_2 \bar{z}$ are contained in some subspace W_3. We may assume that W_3 acts quadratically. Denote by $W_4 = \langle x_1 y_1, \ldots, x_n y_n \rangle$, a diagonal of $W_1 W_2$. We may also assume that W_4 acts quadratically. As $y\overline{y}$, $z\overline{z}$ are in W_4 we get with (1)

$$(2) \qquad 0 = [v, y\overline{y},\ z\overline{z}] = [v, y, \overline{z}] + [v, \overline{y}, z] \text{ for any pair } y, z \in W,\ v \in W.$$

Hence in our situation we have

$$(3) \qquad [v, y_1, \overline{z}] = [v, x_1, z] \text{ and } [v, \overline{y}, y_2] = [v, y, x_2],\ v \in V$$

and

$$(4) \qquad 0 = [v, y_1 \overline{y},\ y_2 \overline{z}] \underset{(1)}{=} [v, y_1, \overline{z}] + [v, \overline{y}, y_2].$$

So by (2) and (3) we have

$$(5) \qquad [v, x_1, z] = [v, y, x_2], v \in V.$$

This gives $[v, x_1 y, x_2 z] = [v, x_1, z] + [v, y, x_2] = 0$ by (5). ∎

LEMMA (0.3) ([S]): *Let G be a Chevalley group over $GF(q)$ and V be an irreducible $GF(2)G$-module. Then $C_V(K)$ is an irreducible $GF(2)P$-module for any parabolic P of G, where $K = O_2(P)$.*

LEMMA (0.4):

(a) Let G be a Chevalley group over $GF(q)$, $P_1, \dots, P_n$ be the set of minimal parabolics of G containing a given Sylow 2-subgroup. Set $K_i = O_2(P_i)$, $i = 1, \dots, n$. Let V be an irreducible module over $GF(q)$. Then V is uniqueley determined by the action of P_i on $C_V(K_i)$, $i = 1, \dots, n$.

(b) If $V = V(\lambda)$, $\lambda = \sum_{i=1}^{n} a_i \lambda_i, \lambda_i$ fundamental weights, then (a) *says that whenever P_i acts nontrivially on $C_V(K_i)$, we get $a_i \neq 0$, otherwise $a_i = 0$.*

(c) Let V be as in (b). If all $C_V(K_i)$ are trivial for all but one $i = i_0$ and $C_V(K_{i_0})$ is a natural module, then $V = V(\lambda)$ for some fundamental weight λ.

Proof: (a) is [RS], while (b) and (c) are easy consequences from (a). ∎

A consequence of (0.4) we will use quite often is

LEMMA (0.5): *Let G be a Chevalley group over $GF(q)$, $G \ncong L_n(q)$, and R be some long root subgroup. Let $Q = O_2(N_G(R))$, $S \in Syl_2(N_G(R))$ and P_i the minimal parabolic with $S \subseteq P_i$, $P_i \nsubseteq N_G(R)$, $O^{2'}(P_i/O_2(P_i)) \cong L_2(q)$. Let V be some irreducible nontrivial module for G over $GF(q)$ with $[V, R, Q] = 0$. Suppose that there is some $g \in G$ such that $R^g \subseteq Q$, $R^g \neq R$ and $(RR^g) \subseteq x^G$ for $x \in R^\sharp$. Then $O^{2'}(P_i)$ has a fixed point on V. Furthermore if $V = V(\lambda), \lambda = \sum_{i=1}^{n} a_i \lambda_i$, λ_i fundamental weights, then the coefficient a_i of λ_i corresponding to P_i is zero.*

Proof: As $G \ncong L_n(q)$, we have that $N_G(R)$ is a maximal parabolic of G. Now we are going to apply (0.1) to $\langle x, x^g \rangle, x \in R^\sharp$. As $C_G(x) = C_G(R)$ we get $G = \langle C_G(x), C_G(x^g) \rangle$. So by (0.1) $0 \neq [V, x] \cap [V, x^g] = U$. Now $\langle Q, Q^g \rangle$ centralizes U and so $S\langle Q, Q^g \rangle = O^{2'}(P_i)$ has a fixed point on V, i.e. $a_i = 0$ by (0.4). ∎

LEMMA (0.6): *Let $G \simeq Sz(q)$, and let V an irreducible $GF(2)G$–module. Then $[V, Z(S), S] \neq 0$, for $S \in Syl_2(G)$.*

Proof: Suppose false. Then $[V, Z(S), Z(S)] = 0$. Any two conjugates of $Z(S)$ generate G. By [BHu;XI,§3] there is some involution $i \in Z(S)$ inverting some element ω of order 5 in G. As $G = \langle Z(S), Z(S)^\omega\rangle = \langle i, Z(S)^\omega\rangle$ we get

$$V = [V, Z(S)] \oplus [V, Z(S)^\omega] = C_V(Z(S)) \oplus C_V(Z(S)^\omega)$$

and

$$C_V(i) \cap C_V(Z(S)^\omega) = 0.$$

So $C_V(i) = C_V(Z(S))$ and $C_V(i) \cap C_V(\omega) = 0$. As i normalizes $\langle\omega\rangle$ we get $C_V(\omega) = 0$. By [Mar] V is the natural module. But then $[V, Z(S), S] \neq 0$, [BHu;XI,§3]. ∎

LEMMA (0.7): *Let $G = G_2(q)$, $q = 2^n$, V be a nontrivial $GF(2)G$–module, R a long root subgroup of G. Let $G_1 = N_G(R)$, G_2 be the other minimal parabolic containing a Sylow 2–subgroup S of G_1. Set $V_i = C_V(O_2(G_i))$. If $[V_2, O^{2'}(G_2)] = 0$ and $[V, R, R] = 0$, then $[V_1, S, S] = 0$.*

Proof: Set $Q = \bigcap_{g \in G_2} O_2(G_1)^g$. Then $|Q| = q^3$. Set $V_3 = C_V(Q)$.

We have $V_1 \subseteq V_3$. As $\langle G_1, G_2\rangle = G$, $V_1 \neq V_3$. We study the action of G_2 on V_3. Let $g \in G$ such that $S = O_2(G_2)R^g$. By assumption $[V_3, R^g, R^g] = 0$. As S/Q is isomorphic to a Sylow 2–subgroup of $L_3(q)$, there is an elementary abelian group W of order q^2 in S/Q such that $S/Q = W(O_2(G_2)/Q)$ and $R^gQ/Q \subseteq W$. As W carries the structure of a 2–dimensional vectorspace over $GF(q)$ and R^gQ/Q corresponds to a one–space, we get with (0.2) that $W \cap O_2(G_2)/Q$ acts quadratically on V_3. Now also $O_2(G_2)/Q$ carries the structure of a 2–dimensional vectorspace over $GF(q)$ and $W \cap O_2(G_2)/Q$ is a one–space. Now any one–space in $O_2(G_2)/Q$ acts quadratically on V_3 as all one–spaces in $O_2(G_2)/Q$ are conjugate under G_2. There is a subgroup T of $O_2(G_2)$ such that TQ/Q is a one–space in $O_2(G_2)/Q$ and $TO_2(G_1) = S$. Hence $[V_1, T, T] \subseteq [V_3, T, T] = 0$ and so $[V_1, S, S] = 0$. ∎

For what follows we fix the following: Let M be some irreducible $GF(2)$-module for $G = G(q)$. Then $M \otimes GF(q)$ is a direct sum of algebraic conjugates of some irreducible $GF(q)$–module V. If $M \otimes GF(q)$ is an algebraic conjugate of a fundamental module for some fundamental weight λ we write $M = V(\lambda)$.

As the assumptions of the theorem and the corollaries are independent of the field, it is enough to prove the theorems for V to be an irreducible $GF(q)$-module, where G is defined over $GF(q)$.

1. Some special modules

Hypothesis 1: Let G be a Chevalley group over $GF(q)$, $q = 2^m$. Let R be some root group and $Q = Z_2(O_2(N_G(R)))$. In case $G \cong Sp(2n, q)$ or $F_4(q)$ we assume in addition R to be long. Let V be a nontrivial irreducible $GF(q)G$–module. Assume $[V, R, Q] = 0$. ∎

In this chapter we always work under hypothesis 1. Remember that if G is not $L(2, q), Sp(4, q)$ or ${}^2F_4(q)$ then $R = Q'$. Furthermore in all cases but ${}^2F_4(q)$ we have $Q = O_2(N_G(R))$.

LEMMA (1.1): *Let H be a maximal parabolic of G such that $[V, R, O_2(H)] = 0$. If there is some $g \in G$ such that $R^g \subseteq H - O_2(H)$ and $[R^g, S] \subseteq O_2(H)$ for some $S \in Syl_2(H)$ and $[\,C\!\!\!\!C_V(O_2(H)), R^g] \neq 0$, then for the minimal parabolic P of G containing S but $P \not\subseteq H$ we get $[\,C\!\!\!\!C_V(S), O^{2'}(P)] = 0$. So if $V = V(\lambda), \lambda = \sum_{i=1}^n a_i\lambda_i$, λ_i fundamental weights, then $a_i = 0$, where $P = P_i$.*

Proof: Set $X = C\!\!\!\!C_V(O_2(H))$. Then by assumption $[R^g, X] \neq 0$. As $[S, R^g] \subseteq O_2(H)$, we have that $[R^g, X]$ is invariant under S. This gives $C\!\!\!\!C_V(S) \cap [R^g, X] \neq 0$. By (0.3) $C\!\!\!\!C_V(S) \subseteq [R^g, X]$. So $C\!\!\!\!C_V(S)$ is centralized by $\langle S, O_2(H)^g\rangle \not\subseteq H$. ∎

LEMMA (1.2) : *Let $G = SL(n, q)$, $n \geq 2$, then $V = V(\lambda)$ for some fundamental weight λ.*

Proof: Let first $n = 2$. Then by hypothesis 1 we have $[V, S, S] = 0$, for some Sylow 2–subgroup S of G. So the assertion follows with [Hig].

The general case we treat by induction on n. We have $n \geq 3$. Let G_1 be the stabilizer of a point and G_{n-1} be the stabilizer of a hyperplane incident to the point for G_1 in the natural representation of G. Then $N_G(R) = G_1 \cap G_{n-1}$. Let $K_i = O_2(G_i)$ and H_i a complement of K_i in $O^{2'}(G_i)$, $i = 1, n-1$. Set $V_i = C\!\!\!\!C_V(K_i)$. As there are conjugates of R^g contained in H_1, H_{n-1} respectively, with $Q^g \cap H_i = Z_2(O_2(N_{H_i}(R^g)))$ we get by induction that V_i are trivial modules or fundamental modules.

Let $V = V(\lambda), \lambda = \sum_{i=1}^{n-1} a_i\lambda_i$. Now as V_1 is trivial or fundamental, we get with (0.4) that for $j > 1$ there is at most one $a_j \neq 0$. Similarly for V_{n-1}. Together we have either just one a_i is nonzero hence 1 and so $V = V(\lambda), \lambda$ a fundamental weight, or $\lambda = a_1\lambda_1 + a_{n-1}\lambda_{n-1}$. In the latter $[V, R, H_1 \cap H_{n-1}] = 0$. Application of (1.1) now yields the contradiction $a_1 = 0 = a_{n-1}$. ∎

LEMMA (1.3): *Let G be $\Omega^+(2n, q)$, $n \geq 4$, then V is the natural or spin module.*

Proof: Let G_1, G_{n-1}, G_n be the three maximal parabolics with connected diagram, i.e. $O^{2'}(G_1/K_1) \simeq \Omega^+(2n-2, q)$, $O^{2'}(G_{n-1}/K_{n-1}) \simeq O^{2'}(G_n/K_n) \simeq SL(n, q)$, where $K_i = O_2(G_i)$. Set $V_i = C_V(K_i)$.

As there are conjugates of R in G_i intersecting K_i trivially, $i = 1, n, n-1$, we may apply (1.2) or use induction. Let $V = V(\lambda)$, then this puts restrictions on the coefficients other than a_i, for $\lambda = \sum_{i=1}^{n} a_i\lambda_i$. So we have that V_1 is either trivial, a natural module or a spin module.

Suppose first that V_1 is trivial. Then by (1.2) both V_{n-1}, V_n are natural modules and so by (0.4) V is the natural module.

So we may assume that V_1 is nontrivial. Let $g \in G$ such that $R^g \leq Q - R$. Then by (0.5) $O^{2'}(P_2)$ has a fixed point on V and $a_2 = 0$. This shows that V_1 has to be a spin module. So we may choose notation such that $a_n = 0$. Then we get with (1.2) that V_n is the natural module and so V_{n-1} is a trivial module. Hence we have that V is a spin module. ∎

LEMMA (1.4): *Let G be $\Omega^-(2n, q)$, $n \geq 3$. Then V is the natural module or the spin module.*

Proof: We first treat the case $n = 3$. Let G_1, G_2 be the two maximal parabolics containing a Sylow 2–subgroup S, where we choose notation such that $G_2 = N_G(R)$. As R acts quadratically and $S = R^g O_2(G_2)$ for some $g \in G$, we get that $V_2 = C_V(O_2(G_2))$ is either a trivial $O^2(G_2/O_2(G_2))$ module or the natural $SL(2, q)$–module.

Suppose first that V_2 is not a trivial module. We may assume $R^g \leq O_2(G_1)$. Now we have $W = [V, R] \cap [V, R^g] \neq 0$. This gives that W is centralized by $\langle Q, Q^g\rangle$. Thus $O^{2'}(G_1) = \langle Q, Q^g\rangle S$ has a fixed point on V, where $S \in Syl_2(G_1)$. So we have that V is the natural module.

Suppose now that $V_2 = [V, R]$ is a trivial module for $O^{2'}(G_2)$. Then

$$[V, R, O_2(G_1)] = 0$$

and so $[V, O_2(G_1), O_2(G_1)] = 0$, as $O_2(G_1)$ is generated by G_1 – conjugates of R. Now there is some $h \in G$ such that $O_2(G_1)(O_2(G_1)^h \cap G_1) = S$. Hence a Sylow 2–subgroup of $SL(2, q^2)$ acts quadratically on V_1, which gives that V_1 is the natural $SL(2, q^2)$–module. So by (0.4) V is the spin module.

So assume now $n \geq 4$. Let G_1 be the maximal parabolic such that $O^{2'}(G_1/K_1) \simeq \Omega^-(2n-2, q), K_1 = O_2(G_1)$. Set $V_1 = C\!\!\!C_V(K_1)$. Then by induction V_1 is trivial, a natural module or a spin module for G_1. Let G_{n-1} be the maximal parabolic containing a Sylow 2–subgroup of G_1 such that $O^{2'}(G_{n-1}/K_{n-1}) \simeq SL(n-1, q)$, $K_{n-1} = O_2(G_{n-1})$. Set $V_{n-1} = C\!\!\!C_V(K_{n-1})$. Then by (0.3) and (1.2) V_{n-1} is either trivial or a fundamental module. In particular (0.4) puts restrictions on the coefficients a_i, $V = V(\lambda)$, $\lambda = \sum_{i=1}^n a_i \lambda_i$.

Suppose first that V_1 is trivial. Then we have that V_{n-1} has to be the natural module. So just a_1 is nonzero and hence V is the natural module by (0.4).

So let V_1 be nontrivial. Let $g \in G$ such that $R^g \leq Q - R$. Then by (0.5) we have $a_2 = 0$. Hence V_1 is the spin module. So $a_i = 0$ for $i > 1$, $i \neq n-1$. If now V_{n-1} is a trivial module we get $a_1 = 0$ and so V is the spin module. So assume that V_{n-1} is nontrivial. Then $a_1 \neq 0$ and so V_{n-1} has to be the natural module. Now by (1.1) $C\!\!\!C_V(S)$ is centralized by $O^{2'}(P_{n-1})$, so $a_{n-1} = 0$, contradicting the fact that V_1 is a spin module. ∎

LEMMA (1.5): *Let G be $Sp(2n, q)$, $n \geq 2$, and R be a long root group. Then V is the natural module or the spin module.*

Proof: Let first $n = 2$. Let $G_1 = N_G(R)$. Then as $S = QR^g \in Syl_2(G_1)$ for suitable $g \in G$, we get that $C\!\!\!C_V(Q)$ is either a trivial or a natural $SL(2, q)$–module for $O^{2'}(G_1/Q) \simeq SL(2, q)$.

If $C\!\!\!C_V(Q)$ is the natural module, we get that $0 \neq [V, R] \cap [V, R^g]$ is centralized by $\langle Q, Q^g \rangle$, and so $C\!\!\!C_V(S)$ is centralized by $S\langle Q, Q^g \rangle = O^{2'}(G_2)$, where G_2 is the other parabolic containing S. Hence V is the natural module by (0.4).

Let $C\!\!\!C_V(Q)$ be a trivial module. Then $[V, R, O_2(G_2)] = 0$ and so $[V, O_2(G_2), O_2 (G_2)] = 0$, as $O_2(G_2)$ is generated by G_2 – conjugates of R. As

$$S = O_2(G_2)(O_2(G_2)^h \cap G_2)$$

for suitable $h \in G$, we get that $V_2 = C_V(O_2(G_2))$ is the natural module for $O^{2'}(G_2/O_2(G_2)) \simeq SL(2,q)$. Then V is the spin module.

Let now $n \geq 3$. Let G_1, G_n be the maximal parabolics containing a given Sylow 2-subgroup S such that

$$O^{2'}(G_1/K_1) \simeq Sp(2n-2,q) \quad \text{and } O^{2'}(G_n/K_n) \simeq SL(n,q),$$

where $K_i = O_2(G_i)$, $i = 1, n$. Set $V_i = C_V(K_i)$, $i = 1, n$. By induction we have that V_1 is a trivial module, a natural module or a spin module. By (1.2) V_n is a fundamental module. Let

$$V = V(\lambda), \quad \lambda = \sum_{i=1}^{n} a_i \lambda_i.$$

If V_1 is trivial we get with (0.4) $a_i = 0$ for $i \neq 1$. So $a_1 \neq 0$ and V is the natural module.

So suppose that V_1 is nontrivial. Choose $R^g \leq Q - R$. Then (0.5) yields $a_2 = 0$. In particular we get that V_1 is a spin module, implying $a_n \neq 0$. If V_n is trivial, we get that V is the spin module. So assume that V_n is the natural module. But as in (1.4) we get that $C_V(S)$ is centralized by $O^{2'}(P_n)$ and so $a_n = 0$, a contradiction. ∎

LEMMA (1.6): *Let G be $SU(n,q)$, or $Sp(2n,q)$, $n \geq 3$. Then $V = V(\lambda)$ for some fundamental weight λ.*

Proof: Suppose first $G \cong SU(n,q)$. Let $n = 3$. Then we have $Q \in Syl_2(G)$. Hence by (0.3) $C_V(Q)$ is an irreducible module for the torus. This gives that $C_V(Q)$ is one dimensional over $GF(q^2)$ and as $[V,R] \leq C_V(Q)$ we get $|[V,R]| \leq q^2$. As there are $g, h \in G$ such that $G = \langle R, R^g, R^h \rangle$, we get $|V| \leq q^6$ and so V has to be the natural module. For $n = 4$, we have $U(4,q) \simeq \Omega^-(6,q)$ and the assertion follows with (1.4).

Assume $n \geq 5$ or G a symplectic group. In the latter the assertion follows for R long with (1.5). So we may assume that R is short. Let $G_1 = N_G(R)$. Then by induction $V_1 = C_V(Q)$ is a trivial module for G_1 or some fundamental module. Let V_1 be nontrivial. There is some $R^g \leq G_1 - Q$ such that $[S, R^g] \leq Q$, where $S \in Syl_2(G_1)$. Now we get with (1.1) that $C_V(S)$ is centralized by $\langle S, Q^g \rangle = O^{2'}(P_1)$. Further as V_1 is fundamental we get with (0.4) that $V = V(\lambda)$, $\lambda = \Sigma a_i \lambda_i$, where $a_i = 0$ for all but one $i, i > 1$. As $C_V(S)$ is centralized by $O^{2'}(P_1)$ we get $a_1 = 0$

and so V is fundamental. If V_1 is trivial, then let G_m be the other maximal parabolic containing S with connected diagram. As $[V, R, \mathcal{C}_G(R)] = 0$, we get $[V, R, O_2(G_m)] = 0$. Now $R \subseteq O_2(G_m)$ and so $[V, \langle R^{G_m}\rangle, \langle R^{G_m}\rangle] = 0$. Hence a Sylow 2–subgroup of $P_1/O_2(P_1)$ acts quadratically on $\mathcal{C}_V(O_2(P_1)) = W$. So W is the natural module by [Hig]. Let $V = V(\lambda)$, $\lambda = \Sigma a_i\lambda_i$. We have by (0.4) $a_1 \neq 0$ and $a_i = 0$ for $i > 1$. So V is the natural module. ∎

LEMMA (1.7): *Let G be $E_n(q)$, $n = 6, 7, 8$, with Dynkin diagram*

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 5 & & n \\ 0 & — & 0 & — & 0 & — & 0 \cdots\cdots 0 & — & 0 \\ & & & & | & & & & \\ & & & & 0 & & & & \\ & & & & 4 & & & & \end{array}$$

Then one of the following holds:

(i) $n = 6$ and $V = V(\lambda_1)$ or $V(\lambda_6)$

(ii) $n = 7$ and $V = V(\lambda_7)$.

Proof:

(i) Let $G = E_6(q)$. Then $N_G(R) = G_4$. Furthermore $O^{2'}(G_4/Q) \simeq SL(6, q)$. So by (1.2) $V_4 = \mathcal{C}_V(Q)$ is trivial or a fundamental module. Let $R^g \leq Q - R$. Then by (0.5) $\mathcal{C}_V(S)$ is centralized by $O^{2'}(P_4)$ for $S \in Syl_2(P_4)$, so if $V = V(\lambda)$, $\lambda = \sum_{i=1}^{6} a_i\lambda_i$, then $a_4 = 0$. Now some a_i, $i \neq 4$, has to be nonzero. We have that $O^{2'}(G_1/K_1) \simeq O^{2'}(G_6/K_6) \simeq \Omega^+(10, q)$, where $K_i = O_2(G_i)$, $i = 1, 6$. Now by (1.3) $V_i = \mathcal{C}_V(G_i)$ is trivial, a natural module or a spin module for $O^{2'}(G_i/K_i)$, $i = 1, 6$. Now V_1 implies that $a_3 = a_5 = 0$ and at most one of a_2 or a_6 is nonzero. Furthermore V_6 implies that $a_2 = a_3 = 0$ and at most one of a_1 or a_5 is nonzero. This shows that $a_2 = a_3 = a_4 = a_5 = 0$. As V_4 is fundamental we get exactly one of a_1 or a_6 is nontrivial, the assertion.

(ii) Let $G \simeq E_7(q)$. Then $G_1 = N_G(R)$. We have $O^{2'}(G_1/Q) \simeq \Omega^+(12, q)$. So by (1.3) $V_1 = \mathcal{C}_V(Q)$ is a trivial, natural or spin module for $\Omega^+(12, q)$. Application of (0.5) shows that $[\mathcal{C}_V(S), O^{2'}(P_1)] = 0$. Let $V = V(\lambda)$, $\lambda = \sum_{i=1}^{7} a_i\lambda_i$, λ_i fundamental weights. We have $a_1 = 0$ by (0.5) and in any case $a_6 = 0$. Further exactly one of a_i, $i \neq 1, 6$, is nonzero. Hence V_1 is nontrivial. As there is some conjugate $R^g \leq G_7 - O_2(G_7)$, we may apply (i). As $a_1 = a_6 = 0$ we get $[\,\mathcal{C}_V(S), O^{2'}(G_7)] = 0$ and so $a_i = 0$ for $i \neq 7$ and then $V = V(\lambda_7)$.

(iii) Let now $G \simeq E_8(q)$. Then $G_8 = N_G(R)$. By (0.5) we again have that $\mathcal{C}_V(S)$ is centralized by $O^{2'}(P_8)$. By (ii) we have that $V_8 = \mathcal{C}_V(Q)$ is a trivial $O^{2'}(G_8)$-module. Let $V = V(\lambda)$, $\lambda = \sum_{i=1}^{8} a_i\lambda_i$, λ_i fundamental weights. By (0.5) $a_8 = 0$. By (ii) $a_i = 0$ for $i \neq 7$. With (1.3) applied to $V_1 = \mathcal{C}_V(O_2(G_1))$ we get $a_7 = 0$, a contradiction. ∎

LEMMA (1.8): *Let G be ${}^2E_6(q)$ or $F_4(q)$ with Dynkin diagram*

$$\begin{array}{ccccccc} 1 & & 2 & & 3 & & 4 \\ 0 & \text{---} & 0 & = & 0 & \text{---} & 0 \end{array}$$

Choose notation such that $N_G(R) = G_1$, then $V = V(\lambda_4)$.

Proof: By (0.5) we have that $[\,\mathcal{C}_V(S), O^{2'}(P_1)] = 0$, $S \in Syl_2(G_1)$. So V_1 is a nontrivial module. Furthermore there is some conjugate $R^g \in G_1 - Q$, such that $R^g \leq Z(O_2(G_4))$. This gives $O^{2'}(G_4) = \langle Q, Q^g\rangle S$. As $V_1 \subseteq [V,R]$ we have $[V,R] \cap [V,R^g] \neq 0$ and so we get that $\mathcal{C}_V(O_2(G_4))$ is a trivial module. Now by (1.5), (1.6) we have that V_1 is the natural module, so $V = V(\lambda_4)$. ∎

LEMMA (1.9): *Let G be $G_2(q)$. Then V is the natural module.*

Proof: The assertion is clear for $q = 2$. So assume $q \geq 4$. Let $G_1 = N_G(R)$ and G_2 be the other maximal parabolic containing a Sylow 2–subgroup S of G. Let $V_i = \mathcal{C}_V(O_2(G_i))$, $i = 1,2$. Then by (0.5) we have that V_2 is a trivial module. By (0.7) we have that $\mathcal{C}_V(Q)$ is the natural G_1/Q–module. Now (0.4) yields the assertion. ∎

LEMMA (1.10): *Let G be ${}^3D_4(q)$. Then V is the natural module.*

Proof: Let G_1, G_2 be the two minimal parabolics containing a Sylow 2–subgroup S of G. Choose notation such that $G_2 = N_G(R)$. Let $K_i = O_2(G_i)$ and $V_i = \mathcal{C}_V(K_i)$, $i = 1,2$. Then by (0.5) we have that V_1 is a trivial module.

Choose conjugates R^g, R^h such that $\langle R, R^g\rangle \trianglelefteq G_1$ $R^h \cap K_1 = 1$, $R^h \leq K_2$. Choose $x \in R$, $y \in R^g$. We have

$$[\,\mathcal{C}_V(xy), x] = [\,\mathcal{C}_V(xy), y]$$

and so

$$|\,C_V(xy):\,C_V(\langle x,y\rangle)| = |\,C_V(xy):\,C_{C_V(xy)}(x)| =$$

$$|[\,C_V(xy),x]\cap[\,C_V(xy),y]| \le |[V,x]\cap[V,y]| \le |[V,R]\cap[V,R^g]|.$$

As $xy \sim x$ we get

$$|\,C_V(x):\,C_V(\langle x,y\rangle| \le |[V,R]\cap[V,R^g]|. \tag{1}$$

Now choose $u \in R^h$. As $O^{2'}(G_1)$ acts 2-transitively on R^{G_1} and $[[V,R]\cap[V,R^g],O^{2'}(G_1)] = 0$, we get that $[V,R]\cap[V,R^g] = [V,(R^g)^u]\cap[V,R^g] = [V,R^g]^u\cap[V,R^g]$. This shows $|[[V,R^g],u]| = |[V,R]:[V,R^g]\cap[V,R]|$. As $R^h \sim R^g$ in G_2, we get for $x \in R$ by (1) using $[V,R^g,x] = 0$:

$$\begin{aligned}|[V,R]\cap[V,R^g]| \ge |\,C_V(x):\,C_V(\langle x,u\rangle| &\ge |[V,R^g]:\,C_{[V,R^g]}(u)|\\ &= |[[V,R^g],u]| = |[V,R]:[V,R^g]\cap[V,R]|.\end{aligned} \tag{2}$$

As $[V,R]\cap[V,R^g]$ is normalized by G_1 and centralized by $O^{2'}(G_1)$, we get that $|[V,R]\cap[V,R^g]| \le |V_1|$. As V_1 is trivial for $O^{2'}(G_1)$ and irreducible for a torus we get $|V_1| \le q^3$ and so $|[V,R]\cap[V,R^g]| \le q^3$. Furthermore $[V,R]$ is a nontrivial module for $L_2(q^3)$ which shows $|[V,R]| \ge q^6$. Hence by (2) $|[V,R]| = q^6$ and $|[V,R]\cap[V,R^g]| = q^3$. If $q \ne 2$, then we have $C_{[V,R]}(S) = C_{[V,R]}(s)$ for any $s \in S^\#$. If $q = 2$ the same is true as any involution in S inverts an element of order 7 in $SL(2,8) \simeq G_2/K_2$. So S acts quadratically on $[V,R]$. Hence V_2 is the natural $SL(2,q^3)$-module. So by (0.4) we get the assertion. ∎

LEMMA (1.11): $G \not\simeq {}^2F_4(q)$.

Proof: Suppose false. The proof follows [MeiStr;(2.17)] where the case of the Tits group has been treated. Let G_1, G_2 be the two minimal parabolics containing a Sylow 2–subgroup S. Choose notation such that $G_1 = N_G(R)$. Let Γ be the coset graph with respect to G_1 and G_2. Let $(1,2,3,4,5)$ be a path of length 4 in Γ with "$G_1 = G_1$" and "$G_2 = G_2$". Set $Z_2 = \langle R^{G_2}\rangle$ and $U_1 = \langle Z_2^{G_1}\rangle$, then $U_1 = Q$. We know that $[V,R,U_1] = 0$. As $U_1 \not\le O_2(G_3) \not\ge U_5$ we get $\langle U_1,U_5\rangle O_2(G_3) \ge O^2(G_3)$. As $U_1\cap U_5$ is centralized by $\langle U_1,U_5\rangle$, we see $|U_1\cap U_5| \le q$. Now $|U_1\cap O_2(G_3)| = q^4$, which gives $U_1\cap O_2(G_3) \not\le U_3$. So we get $\langle U_1,U_5\rangle \ge O^2(G_3)$ and $(O^2(G_3)\cap G_1)R = G_1\cap G_2$. Moreover $\langle U_1,U_5\rangle$

centralizes $[\mathbb{C}_V(U_1), Z(O_2(G_5))]$ and hence $G_1 \cap G_2$ centralizes $[U, Z(O_2(G_5))]$ for any chief factor U of G_1 in $\mathbb{C}_V(U_1)$. As $O^{2'}(G_1/O_2(N_G(R))) \simeq Sz(q)$ we get with (0.6) that $[U, O^{2'}(G_1)] = 1$. Hence we have that $[\mathbb{C}_V(Q), O^{2'}(G_1)] = 1$.

Choose now $R^g \leq Q - R$. Then (0.5) shows that $\mathbb{C}_V(S)$ is centralized by $O^{2'}(G_2)$. But then $\mathbb{C}_V(S)$ is centralized by G, a contradiction. ∎

Now (1.1) – (1.11) and (0.6) prove the Theorem.

2. Strong quadratic modules

In this chapter we are going to prove Corollary 1. So we assume

Hypothesis 2: Let G be a Chevalley group over $GF(q)$, $q = 2^n$, V a nontrivial irreducible $GF(q)G$-module and A be a nontrivial 2–subgroup of G such that $[V, A, A^g] = 0$ for any $g \in G$ with $[A, A^g] = 1$. ∎

Choose A maximal satisfying hypothesis 2. Let $h \in G$ such that $[A, A^h] = 1$. Set $B = \langle A, A^h \rangle$. Then for $g \in G$ with $[B, B^g] = 1$ we get $[V, \langle A, A^h \rangle, \langle A^g, A^{hg} \rangle] = 0$. By maximality of A this implies $A = B$, i.e. A is weakly closed in $\mathbb{C}_G(A)$ with respect to G.

For the remainder of this chapter let G, V be as in hypothesis 2 and $A \leq G$ be maximal satisfying hypothesis 2.

LEMMA (2.1): *Let $G \not\simeq Sp(4, q)$, $G_2(2)$ and R be a long (if $G \simeq F_4(q)$ any) root group. Suppose there is some quadratic fours group W, (with $\langle W^{N_G(R)} \rangle$ nonabelian in case $G \simeq SL(n, q)$), contained in $N_G(R)$ such that one of the following holds*

(i) $|W \cap Z(O_2(N_G(R)))| = |W \cap R| = 2$.

(ii) $W \cap Z(O_2(N_G(R))) = 1$ *and there is some* $g \in G$ *such that* $w_1^g = w_1$, $w_2^g = w_2 r$, *for* $1 \neq r \in R$ *and* $W = \langle w_1, w_2 \rangle$.

Then (G, V) satisfies the assertion of corollary 1.

Proof: Suppose (ii), then $\langle w_1, r \rangle$ satisfies (i). So we may assume that we have (i). Set $Q = Z_2(O_2(N_G(R)))$. Now we get $\langle W^{\mathbb{C}_G(R \cap W)} \rangle \geq Q$ and as $[V, R \cap W, W^{\mathbb{C}_G(R \cap W)}] = 0$ we get $[V, R \cap W, Q] = 0$. Now $[V, R, Q] = [V, (R \cap W)^{N_G(R)}, Q^{N_G(R)}] = 0$. Application of the theorem gives that (G, V) is as in the corollary 1.

LEMMA (2.2): *Let $G \not\simeq SL(n,q), U(3,q), Sp(4,q), Sz(q)$ or $G_2(2)$. Let R be a long root group. If $A \cap R \neq 1$, then corollary 1 holds.*

Proof: As $G \not\cong SL(2,q)$, $U_3(q)$, or $Sz(q)$, we have R is not weakly closed in $\mathbb{C}_G(R)$. So $A \not\leq R$. If $A \not\leq Z(O_2(N_G(R)))$, then A contains a fours group satisfying (2.1)(i). (Notice $G \not\cong SL(n,q)$). So (2.1) yields the assertion. Let $A \subseteq Z(O_2(N_G(R))$. So we have $Z(O_2(N_G(R))) \neq R$ and then $G \simeq Sp(2n,q)$ or $F_4(q)$. As $N_G(R)$ acts indecomposably on $Z(O_2(N_G(R)))$ and A is weakly closed in $\mathbb{C}_G(A)$ we get $A = Z(O_2(N_G(R)))$. In case of $F_4(q)$ there is another root group L such that $L \cap A \neq 1$. Now the same argument shows $A = Z(O_2(N_G(L)))$. As $Z(O_2(N_G(R))) \neq Z(O_2(N_G(L)))$ in $F_4(q)$ we get a contradiction.

We are left with $G \simeq Sp(2n,q)$, $n \geq 3$, $|A| = q^3$. Let L be a short root group $L \leq A$. Then $\langle A^{N_G(L)} \rangle = O^2(N_G(L))$. This gives $[V, L, O^{2'}(N_G(L))] = 0$. By (1.6) we get that V is the natural module and we are done. ∎

For the remainder of this chapter we fix the following notation: Let R be a long root group centralized by A and $Q = O_2(N_G(R))$. Let S be a Sylow 2-subgroup of $N_G(R)$ containing A.

LEMMA (2.3): *If $G \not\simeq (S)L(n,q), {}^2F_4(q)$, $Sp(2n,q)$, $F_4(q)$, $(S)U(3,q)$, $Sz(q)$ or $G_2(2)$, then corollary 1 holds.*

Proof: Let $A \leq Q$. In all Chevalley groups not excluded by assumption we have $[A,Q] \leq R$ and if $a \in A - R$, then $[a,Q] = R$. So as A is weakly closed in $\mathbb{C}_G(A)$, we get either $A \subseteq R$ or $R \leq A$. Now the assertion follows with (2.2).

Let now $A \not\leq Q$. Suppose $A \cap Q = 1$. Then $[N_Q(A), A] = 1$. As $\mathbb{C}_G(Q) \subseteq Q$, we have $N_Q(A) \neq Q$. Now for $x \in Q - N_Q(A), [x,A] \subseteq N_Q(A)$, we have $[A^x, A] = 1$. But A is weakly closed in $\mathbb{C}_G(A)$, a contradiction. So $A \cap Q \neq 1$.

Let $1 \neq x \in A \cap Q$. Set $Y = \mathbb{C}_Q(x)$. Choose $a \in A^\sharp$, $g \in Y$ with $a^g = ay$, $y \neq 1$. Then $\langle x, y \rangle$ acts quadratically on V. If $x \in R$, then (2.2) yields the assertion. Suppose $x \notin R$. If $y \notin Z(Y)$, then $\langle x,y \rangle \cap Z(O_2(N_G(R))) = 1$. Furthermore there is some $u \in Y$ with $y^u = yr$, $r \in R^\sharp$, as $R = O_2(N_G(R))'$. So (2.1)(ii) applies and corollary 1 holds. We are left with $y \in Z(Y)$. Let $Y_1 = [a, Y] \subseteq Z(Y)$. We have $|Q : Y| = q$ and $|Q : \mathbb{C}_Q(t)| = q$ for any $t \in Q - Z(Q)$. If $|Y_1| > q$, then there is some $t \in Q - Y$ with $x^t = xr$, $r \in R^\sharp$, but $y_1^t = y_1$ for some $y_1 \in Y_1^\sharp$. So either (2.1)(ii) applies to $\langle y_1, x \rangle$ or $y_1 \in R$ and so (2.1)(i) applies as $x \notin R$. In any case we get the assertion.

So as $[Q,A] \le Y$, we get $|[Q/R,a]| \le q^2$ for any $a \in A - Q$. In particular $G \not\cong E_n(q)$, $n = 6,7,8$, ${}^2E_6(q)$, or ${}^3D_4(q)$, by using Chevalley commutator formulas. So we have $G \cong (S)U(n,q)$, $G_2(q)$ or $\Omega^{\pm}(2n,q)$.

Let $G \simeq (S)U(n,q)$. Then Q/R is the natural module for $SU(n-2,q)$. So A is a group of transvections on the natural module implying $|A : A \cap Q| \le q$ and there is some $g \in G$ such that $AQ \le QR^g$. Furthermore by (2.1) we may assume $A \cap Q \le Z(Y)$, otherwise $\langle x,y \rangle, x \in A \cap Q, y \notin Z(Y)$ satisfies (2.1)(ii).

We have $[Z(Y), R^g] = 1$. Let $a \in A - Q$, then $a = uv$, $u \in Y$, $v \in R^g$, $[u,v] = 1$. Suppose $u \notin Z(C_Q(R^g))$. Then there is some $s \in C_Q(R^g)$ with $u^s = ur$, $r \in R^\sharp$. Now $a^s = ar$ but $x^s = x$. Hence $\langle x,a \rangle$ satisfies (2.1)(ii) and so we have the assertion. Assume now $u \in Z(C_Q(R^g))$. Then uv is in a conjugate of R^g and so we have the assertion with (2.2).

Let $G \cong G_2(q)$ and $L \ne N_G(R)$ be the other parabolic containing S. Set $E = O_2(L)$. As $[E, O^{2'}(L)] \ne 1$, we get as before that $A \cap E \ne 1$. As $Z(E)^\sharp \subseteq R^G$, we get with (2.2) that $A \cap Z(E) = 1$ or the conclusion holds. So assume $A \cap Z(E) = 1$. Then $[A, Z(E)] = 1$ and so $A \subseteq E$. As $E' = Z(E)$ this contradicts the fact that A is weakly closed in $C_G(A)$.

So assume finally that $G \cong \Omega^{\pm}(2n,q)$. Let L be the parabolic with $O^{2'}(L/E) \cong \Omega^{\pm}(2n-2,q)$, $E = O_2(L)$. Again $A \cap E \ne 1$. If $A \subseteq E$, then by weak closure $A = E$, which with (2.2) yields the assertion. Let $A \not\subseteq E$. As E is the natural module for $\Omega^{\pm}(2n-2,q)$, there is some $t \in A - E$ and $s \in E$ such that $[t,s] = f$ is contained in a some long root group F. So $\langle f,e \rangle, e \in E \cap A$ is a quadratic fours group. As $e \notin T$ and $Z(O_2(N_G(F))) = F$, $\langle f,e \rangle$ satisfies (2.1)(i). So we get the assertion. ∎

In what follows we analyze the cases left in (2.3) one by one.

LEMMA (2.4): *If $G \cong Sp(2n,q)$, corollary 1 holds.*

Proof: Suppose false. Let L be a short root subgroup contained in $Z(S)$ and $M = O_2(N_G(L))$. Suppose first $n = 2$. Obviously we may assume $q \ne 2$. Then $A \le Q$ or M as any elementary abelian subgroup of S is in Q or M. As $N_G(Q)$ and $N_G(L)$ act irreducibly on Q, M, respectively, we get $A = Q$ or $A = M$ by weak closure. Now the assertion follows with the theorem.

So we have $n \ge 3$. Let $A \le Q$. If $A \cap R \ne 1$ the assertion follows with (2.2). So we may assume $A \cap R = 1$. Now $[A,Q] \subseteq R$. If $A \not\subseteq Z(Q)$, then $\langle A^Q \rangle = \langle A, R \rangle$, contradicting $A^Q = A$ and $R \cap A = 1$. So $A \subseteq Z(Q)$. If $q > 2$, then $N_G(Q)$ acts indecomposably on $Z(Q)$. As A is weakly closed in $Z(Q)$, we get $A = Z(Q)$, contradicting $A \cap R = 1$. So we have $q = 2$ and A is a complement of R in $Z(Q)$, i.e. $|A| = 4$. Let U be the maximal parabolic containing S such that $O^{2'}(U/O_2(U)) \cong L_n(2)$. Then $\langle A^U \rangle$ is the natural module ($n \geq 3$), a contradiction to the fact that A is weakly closed in $\mathcal{C}_G(A)$.

So we may assume $A \not\leq Q$. We have $O^{2'}(N_G(R)/Q) \cong Sp(2n-4,q) \times L_2(q)$. By (2.2) we may assume $A \cap R = 1$. Suppose $A \cap Z(Q) \neq 1$. Let $a \in A \cap Z(Q)^\sharp, x \in A - Q$. Then $\langle a, x \rangle$ acts quadratically on V. So $[V, a, x^{\mathcal{C}_{N_G(R)}(a)}] = 0$. Hence $[V, a, Q] = 0$ because any normal subgroup of $\mathcal{C}_G(Z(Q))$ not contained in Q has to contain Q. So there is a fours group $\langle ba, a \rangle, ba \in R$ acting quadratically on V. Let $a \in L$. If $A \leq M$, then by weak closure $A = M$. But then $A \cap R \neq 1$, a contradiction. So we have $A \not\leq M$. But as $A \cap M \not\subseteq L$ by weak closure, we get some $t \in M \cap A, t$ in a long root group, which with (2.2) yields the assertion.

So we are left with $a \notin L$. Then a is an involution of type c_2. So we may assume $b \in L$. As $a \in N_G(L)$ we get $a^{\mathcal{C}_G(b)} \geq M$ and so $[V, b, M] = 0$. Now $[V, b, M \cap Q] = 0$. As $[V, a, Q] = 0$, we get $[V, ab, M \cap Q] = 0$. But then $[V, ab, (M \cap Q)^{\mathcal{C}_G(ab)}] = 0$ and so $[V, ab, Q] = 0$, and the assertion holds by the theorem.

So we just have to treat the case $A \cap Z(Q) = 1$. Now weak closure implies $A \subseteq \mathcal{C}(Z(Q))$. As in (2.3) we get that $|[Q/Z(Q), t]| \leq q^2$ for any $t \in A - Q$. This shows $|A : A \cap Q| \leq q$ and A induces transvections on both $Sp(2n-4, q)$-modules involved in $Q/Z(Q)$. Now there is some $\rho \in N_G(R)$, $o(\rho) = q + 1$, with $[A, \rho] \subseteq Q$. As $A \subseteq N(M)$, we get $A \subseteq N(M^\rho)$ and so $A \cap M \neq 1 \neq A \cap M^\rho$. But $M \cap M^\rho = (Q \cap M) \cap (Q \cap M^\rho) \subseteq Z(Q)$. This shows that there is some $\langle x, y \rangle \subseteq A$, $\langle x, y \rangle \cap Z(Q) = 1$, $x \in M$, $y \in M^\rho$. Hence there is some $g \in M \cap Q$ with $y^g = yr$, $r \in R^\sharp$. Application of (2.1)(ii) gives the assertion. ∎

LEMMA (2.5): *If $G \cong F_4(q)$, corollary 1 holds.*

Proof: Let $A \leq Q$. If $A \not\leq Z(Q)$, then $\langle A^Q \rangle = AR$. As A is weakly closed in $\mathcal{C}_G(A)$ we get $R \subseteq A$. Now (2.2) yields the assertion. So assume $A \subseteq Z(Q)$. By weak closure we get $A = Z(Q)$ and so again $R \leq A$ and (2.2) yields the assertion. So we may assume $A \not\leq Q$. Now we show that we may assume that A induces transvections on $Z(Q)$. For this choose $t \in A - Q$ such that $|[Z(Q), t]| \geq q^2$. As $A \cap Z(Q) \neq 1$, there is a foursgroup $\langle a, b \rangle \leq Z(Q)$, $a \in A$, such that $\langle a, b \rangle$ acts quadratically on V, b is a root element, and $\langle a,b\rangle \leq O_2(C_G(b)) - Z(C_G(b))$. But now (2.1)(i) yields the assertion.

So assume that A induces transvections on $Z(Q)$. This shows $|A : A\cap Q| \leq q$ and $|[A, Q/Z(Q)]| = q^4$. If $|A\cap Q : A\cap Z(Q)| > q$, then there is some foursgroup $\langle x,y\rangle \subseteq A\cap Q$ such that $\langle x,y\rangle \cap Z(Q) = 1$ and for some $h \in Q$, $x^h = x$, $y^h = yr$, $r \in R^\sharp$, as $|Q : C_Q(t)| = q$ for any $t \in Q - Z(Q)$. Now (2.1)(ii) yields the assertion. So assume $|A\cap Q : A\cap Z(Q)| \leq q$. Let $a \in A\cap Q - Z(Q)$. Then the action of A on Q shows that there is some $g \in C_Q(a) - Z(Q)$, such that $b^g = bc$, for $b \in A - Q$ and c not in the root group determined by a. But then $\langle a,c\rangle$ acts quadratically on V, and (2.1)(i) yields the assertion. So assume $A\cap Q = A\cap Z(Q)$. Let L be the other type of root groups such that $LR \subseteq [Z(Q), A]$. Set $M = O_2(N_G(L))$. Then as A induces transvections on $Z(Q)$, we get $AQ \subseteq Z(M)Q$. Now $(Z(M)Q)' \subseteq Z(M)$, so $[A,Q] \subseteq Z(M)$. As $|[a, Q/Z(Q)]| = q^4$ for $a \in A - Q$, we get that $|[a,Q]L/L| = q^5$ and $[a,Q,a] = 1$. But then $[a,Q]L/L$ is centralized by a and $[[A,Q],A] = 1$. Hence $\langle A^Q\rangle$ is abelian. Now weak closure gives $|A\cap Z(M)| \geq q^5$. But then as $|Z(M) : Z(M)\cap Q| = q$, we get $|A\cap Q| = |A\cap Z(Q)| \geq q^4$. So $|A\cap Z(Q)\cap Z(M)| \geq q^4$, but this contradicts $|Z(Q)\cap Z(M)| = q^2$. ∎

LEMMA (2.6): *If $G \cong SL(n,q)$, corollary 1 holds.*

Proof: Let $Q = E_1E_2$, $E_i \trianglelefteq N_G(R)$, E_i elementary abelian, $i = 1,2$. Then $A \leq N_G(E_1)$. As in all cases before we get $A\cap E_1 \neq 1$ by weak closure. But all elements in $E_1^\sharp$ are conjugate and so we may assume $A\cap R \neq 1$.

If $A \subseteq E_1$ we get by weak closure $A = E_1$ and then the assertion with [MeiStr1;(1.6)]. So assume $A \not\subseteq E_1$. Choose $x \in R\cap A$, $y \in A - E_1$. Then $\langle x,y\rangle$ is a quadratic fours group and $U = \langle\langle x,y\rangle^{N_G(R)}\rangle \geq E_1$. So $U' \neq 1$. Now we may apply (2.1) (i) to $\langle x,y\rangle$ which implies the assertion. ∎

LEMMA (2.7): $G \not\cong {}^2F_4(q)$.

Proof: Suppose false. By (2.2) $A\cap R = 1$. So by weak closure we have $A \not\leq O_2(N_G(R))$. Let $t \in A - O_2(N_G(R))$. Then $|[t,Q]| = q^2$. As A is weakly closed in $C(A)$ we get that either $|A\cap Q| \geq q^2$ or there is some $a \in A\cap Q$, $b \in Q$ such that $t^b = tr$, where r is not in the root group belonging to a. Hence $\langle a,r\rangle$ is a quadratic fours group. So in both cases we get a quadratic fours group in Q which is not contained in a root subgroup, which contradicts (2.1). ∎

LEMMA (2.8): *If $G \cong Sz(q)$, then V is the natural module.*

Proof: We have $A = R$. As $G = \langle R, R^g \rangle$ for any $g \in G - N_G(R)$, we get $V = [V,R] \oplus [V,R^g]$. By [Mar] V is the natural module. ∎

LEMMA (2.9): *If $G \cong (S)U(3,q)$, then V is a basic module as in corollary 1.*

Proof: We have $R = A$. Let $X = \langle R, R^g \rangle \simeq SL(2,q)$ for some $g \in G$. Then we know that V involves just natural X - modules by (2.6). As any irreducible G-module is a restriction of some irreducible $SL(3,q^2)$-module, we may assume that V is irreducible for $H = (S)L(3,q)$ and $X \leq G \leq H$. By Steinberg's tensor product lemma any irreducible H-module is a tensorproduct of algebraic conjugates of the basic modules. We may assume that all this basic modules are nontrivial. Then V involves tensorproducts of algebraic conjugates of the natural X-module. The quadratic action of X now gives that the tensor product just has one factor, i.e. V is an algebraic conjugate of a basic module. Over $GF(2)$ this means that V is a basic module. ∎

LEMMA (2.10): *If $G \cong G_2(2)$, then V is the natural module.*

Proof: We have a quadratic fours group. It is easy to see that V has to be the natural 6-dimensional module. ∎

Now Corollary 1 follows from (2.3) - (2.10).

3. Quadratic fours groups

Hypothesis 3: Let G be a Chevalley group over $GF(q)$, $q = 2^n$, V a nontrivial irreducible $GF(q)G$-module. Let R be a long root subgroup of G and E be a fours group with $|E \cap R| = 2$. Assume $[V,E,E] = 0$. ∎

For the remainder of this section we are working under hypothesis 3. We are going to prove Corollary 2.

We fix the following notation:

(i) $\langle e \rangle = E \cap R$.

(ii) $X = \langle E^{\mathcal{C}_G(e)} \rangle$.

LEMMA (3.1): *If $X \supseteq Z_2(O_2(N_G(R)))$, then the conclusion of corollary 2 holds.*

Proof: As $[V, e, X] = 0$ and $\langle e^{N_G(R)} \rangle = R$, the assertion follows from the main theorem. ∎

LEMMA (3.2): *If $G \not\cong SL(n,q)$, $Sp(2n,q)$, $F_4(q)$ or $G_2(q)$ the conclusion of corollary 2 holds.*

Proof: In all other cases $C_G(R)$ acts irreducibly on $Z_2(O_2(N_G(R)))/R$ and $Z_2(O_2(N_G(R)))/R$ is the only minimal normal subgroup of $C_G(R)/R$. So $Z_2(O_2(N_G(R))) \subseteq X$. The assertion follows with (3.1). ∎

LEMMA (3.3): *Let $G \cong SL(n,q)$, then $V = V(\lambda)$ for some fundamental weight λ.*

Proof: By (3.1) we may assume $O_2(N_G(R)) \not\subseteq X$. Let $O_2(N_G(R)) = E_1E_2$, E_i elementary abelian of order q^{n-1}, $E_i \triangleleft N_G(R)$, $i = 1, 2$. Then we may assume $X \subseteq E_1$. Assume first $X = E_1$. As $X = \langle e^{N_G(x)} \rangle$, we get $0 = [V, \langle e^{N_G(R)} \rangle, X] = [V, X, X]$. As X is weakly closed in $C_G(X)$ with respect to G, the assertion follows with corollary 1.

So $X \neq E_1$. This is only possible for $n = 3$. We have that R acts quadratically on V. Furthermore $S = E_1R^g \in Syl_2(N_G(E_1))$ for suitable $g \in G$. Set $V_1 = C_V(E_1)$.

If $[R^g, V_1] \neq 0$, then $C_V(S)$ is centralized by $\langle S, X^g \rangle$. As $X^g \subseteq E_1^g$ and $E_1^g \cap N_G(E_1) = R$, we get $\langle S, X^g \rangle = O^{2'}(N_G(E_2))$. As by [Hig] V_1 is a natural $SL(2,q)$-module, we get with (0.4) that V is the natural module, as V_1, V_2 are like in the natural module.

If $[V_1, O^{2'}(N_G(E_1))] = 0$, then $C_V(E_2)$ has to be a natural module by [Hig] as $S = E_2R^h$ for suitable $h \in G$, $S \in Syl_2(N_G(E_2))$. Now again by (0.3) V is a natural module. ∎

LEMMA (3.4): *Let $G \cong Sp(2n,q)$, then $V = V(\lambda)$ for some fundamental weight λ.*

Proof: By (3.1) we may assume $X = Z(O_2(N_G(R)))$. Let $f \in X$, $f \in \tilde{R}$, $\tilde{R}$ a short root subgroup. Then $[V, \langle e, f \rangle, \langle e, f \rangle] = 0$. Set $Y = \langle \langle e, f \rangle^{C_G(f)} \rangle$. As

$e \in O_2(\mathit{C}_G(\tilde{R}))$, we get $Y = O_2(\mathit{C}_G(\tilde{R}))$. Furthermore $\langle f^{N_G(\tilde{R})}\rangle = \tilde{R}$ and so $[V, \tilde{R}, Y] = 0$. Now the assertion follows with (1.6). ∎

LEMMA (3.5): *Let $G \cong F_4(q)$, then $V = V(\lambda)$ for some fundamental weight λ.*

Proof: By (3.1) we may assume that $X = Z(O_2(N_G(R)))$. So we may assume $E = \langle e, f\rangle$, $e \in R$, $f \in \tilde{R}$, $\tilde{R} \subseteq X$, $\tilde{R}$ a short root subgroup. As G possesses an outer automorphism α with $R^\alpha = \tilde{R}$, we may also assume that $\langle E^{\mathit{C}_G(f)}\rangle = Z(O_2(N_G(\tilde{R})))$. So $E \subseteq Z(O_2(N_G(\tilde{R}))) \cap Z(O_2(N_G(R))) = R\tilde{R}$. Furthermore we have $\langle e^{N_G(R)}\rangle = R$ and $\langle f^{N_G(\tilde{R})}\rangle = \tilde{R}$. This gives

(∗) $[V, R, Z(O_2(N_G(R)))] = 0 = [V, \tilde{R}, Z(O_2(N_G(\tilde{R})))]$.

Now we fix a Sylow 2-subgroup S of $N_G(R)$ with $Z(S) = R\tilde{R}$. Let G_1, G_2, G_3, G_4 be the four maximal parabolic subgroups of G containing S. Set $K_i = O_2(G_i)$ and $V_i = \mathit{C}_V(K_i)$.

Let V_i be a nontrivial module for $O^{2'}(G_i)$ for some i. There is some $g \in G$ such that for $T = R^g$ or $\tilde{R}^g$ the following holds:

(i) $T \subseteq G_i \backslash O_2(G_i)$.

(ii) $[T, S] \subseteq O_2(G_i)$.

(iii) $[V_i, T] \neq 0$.

By (0.3), (ii) and (iii) $\mathit{C}_V(S) \subseteq [V_i, T]$ and so by (∗) $\mathit{C}_V(S)$ is centralized by $\langle S, Z(O_2(N_G(T)))\rangle$. As $Z(O_2(N_G(T))) \not\subseteq G_i$, we get that $\mathit{C}_V(S)$ is centralized by $O^{2'}(P_i)$, where P_i is the minimal parabolic subgroup containing S but not contained in G_i. So we have

(∗∗) If $[V_i, O^{2'}(G_i)] \neq 0$ for some $i \in \{1,2,3,4\}$, then $[\mathit{C}_V(S), O^{2'}(P_i)] = 0$, where $S \subseteq P_i$, $P_i \not\subseteq G_i$.

From (∗∗), we get that there is exactly one $j \in \{1,2,3,4\}$ such that $[\mathit{C}_V(S), O^{2'}(P_j)] \neq 0$. As there is some $h \in G$ such that for $U = R^h$ or $\tilde{R}^h$ we have $S = O_2(P_j)U$ and $[V, U, U] = 0$ by (∗), we get with [Hig] and (0.3) that $\mathit{C}_V(O_2(P_j))$ is the natural module for $O^{2'}(P_j/O_2(P_j)) \cong SL(2, q)$. Now $V = V(\lambda_j)$ by (0.4). ∎

LEMMA (3.6): *Let $G \cong G_2(q)$, then V is the natural module.*

Proof: By (3.1) we may assume $X \subseteq O_2(N_G(R))$, $X \neq O_2(N_G(R))$. Let G_1, G_2 be the two maximal parabolics containing $S \in Syl_2(N_G(R))$, where $G_1 = N_G(R)$. There is some $g \in G$ such that $R^g O_2(G_2) = S$. As $R \subseteq X$, we have $[V, R, R] = 0$.

Suppose first that $[\,\mathcal{C}_V(O_2(G_2)),\ O^{2'}(G_2)] \neq 0$. Then $\mathcal{C}_V(S) \subseteq [\,\mathcal{C}_V(O_2(G_2)), R^g]$ and so $\mathcal{C}_V(S)$ is centralized by S and X^g. But $X^g \not\subseteq G_2$ and so $\langle S, X^g\rangle = O^{2'}(G_1)$ centralizes $\mathcal{C}_V(S)$. By (0.3) and [Hig] $\mathcal{C}_V(O_2(G_2))$ is the natural $O^{2'}(G_2)/O_2(G_2) \cong SL(2,q)$ module. So by (0.4)(c) $V = V(\lambda_2)$.

Let $f \in Z(O_2(G_2)) - R$, then $f \sim e \sim ef$. By (0.1) $W = [V, f] \cap [V, e] \neq 0$. Let $g \in G_2$ with $e^g = f$, then W is centralized by $\langle X, X^g\rangle$.

We have that W is invariant under $O_2(G_2)$ and so $W \cap \mathcal{C}_V(O_2(G_2)) \neq 0$. This shows that there is some $v \in \mathcal{C}_V(O_2(G_2))^\sharp$ such that v is centralized by some element $\nu \neq 1$ of odd order in $O^{2'}(G_2)$. (Notice that $\langle X^{N_G(S)}\rangle = O_2(G_1)$ and so $X \not\subseteq O^2(G_2)$). But $\mathcal{C}_V(O_2(G_2))$ is the natural module, a contradiction.

So we have that $[\,\mathcal{C}_V(S), O^{2'}(G_2)] = 0$. As $R \subseteq X$, we get with (0.6) that $\mathcal{C}_V(O_2(G_1))$ is the natural module for $O^{2'}(G_1)/O_2(G_1)$ and so by (0.3) V is the natural module for G. ∎

Now Corollary 2 follows from (3.1) – (3.6).

References

[BHu] N. Blackburn and B. Huppert, *Finite Groups III*, Grundlehren der mathematischen Wissenschaften 243, Springer, Berlin, 1982.

[Hig] G. Higman, *Odd characterizations of finite simple groups*, Univ. of Michigan Lecture Notes, 1968.

[Ho] C. Y. Ho, *On the quadratic pairs*, J. Algebra **43** (1976), 338–358.

[Mar] R. Marineau, *On 2-modular representations of the Suzuki groups*, Amer. J. Math. **94** (1972), 55–72.

[Mei] U. Meierfrankenfeld, *Quadratic modules in odd characteristic*, in preparation.

[MeiStr1] U. Meierfrankenfeld and G. Stroth, *Quadratic GF(2)-modules for sporadic groups and alternating groups*, Comm. in Algebra **18** (1990), 2099–2140.

[MeiStr2] U. Meierfrankenfeld and G. Stroth, *On quadratic GF(2)-modules for Chevalley groups over fields of odd order*, Arch. Math. **55** (1990), 105–110.

[PS] A. A. Premet and I. D. Suprunenko, *Quadratic modules for Chevalley groups over fields of odd characteristics,* Math. Nachr. **110** (1983), 65–96.

[RS] M. Ronan and S. Smith, *Sheaves on buildings and modular representations of Chevalley group,* J. Algebra **96** (1985), 319–346.

[S] S. Smith, *Irreducible modules and parabolic subgroups,* J. Algebra **75** (1982), 286–289.

[Th] J. G. Thompson, *Quadratic pairs,* Actes Congrés Intern. Math. **1** (1970), 375–376.

[Ti] F. Timmesfeld, *Groups generated by k-root subgroups,* Invent. Math. **106** (1991), 575–666.